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ORIGINAL ARTICLE

On curvatures and points of the translation surfaces in Euclidean 3-space

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Abstract In this paper, translation surfaces generated by two arbitrary space curves in 3-dimensional Euclidean space have been investigated. Furthermore, a classification of some special points on these surfaces have been given. Moreover, we have obtained the associated geometric attributes for these surfaces, e.g. the Gaussian curvature and the mean curvature. Finally, some important results and examples that show the idea were presented.

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1. Introduction

The famous type of a translation surface is generated by two planar curves lying on orthogonal planes. This type will be called as a translation surface of plane type and takes the form:

$$X(u,v) = (u,0,f(u)) + (0,v,g(v)), \tag{1}$$

where $f(u)$ and $g(v)$ being smooth functions of the variables u and v, respectively.

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The translation surfaces of plane type have been investigated from the various viewpoints by many differential geometers. Verstraelen et al. [1] have investigated minimal translation surfaces of plane type in n-dimensional Euclidean spaces. Liu [2] has given the classification of the translation surfaces of plane type with constant mean curvature or constant Gauss curvature in 3-dimensional Euclidean space $\mathbf{E}^3$ and 3-dimensional Minkowski space $\mathbf{E}^3_1$. Yoon [3] has studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map $\mathbf{G}$ satisfies the condition $\Delta\mathbf{G} = \mathbf{AG}, \mathbf{A} \in Mat(3,R)$ where Δ denotes the Laplacian of the surface with respect to the induced metric and $Mat(3,R)$ the set of 3×3 real matrices. Dillen et al. [4] have derived a classification of translation surfaces in the 3-dimensional Euclidean and Minkowski space, satisfying the Weingarten condition. Yoon [5] has classified a polynomial translation surfaces in Euclidean 3-space satisfying the Jacobi condition with respect to the Gaussian curvature, the mean curvature and the second Gaussian curvature. Munteanu and Nistor [6] have studied

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the second fundamental form of translation surfaces of plane type in $\mathbf{E}^3$. They have given a non-existence result for polynomial translation surface with vanishing second Gaussian curvature. Bekkar and Senoussi [7] have studied the translation surfaces in the 3-dimensional Euclidean and Lorentz-Minkowski space under the condition $\Delta^{III} r_i = \mu_i r_i, \mu_i \in R$, where Δ^{III} denotes the Laplacian of the surface with respect to the third fundamental form III. They shown that in both spaces a translation surface satisfying the preceding relation is a surface of Scherk. Cetin et al. [8,9] have investigated the translation surfaces according to Frenet frames in Euclidean and Minkowski 3-space. They have given some properties of these surfaces using non-planer space curves.

The general form of translation surface is the surface that can be generated from two arbitrary space curves by translating either of them parallel to itself. In such a way that each of its points describes a curve that is a translation of the other curve. A generalized type of a translation surface parameterized by:

$$X(u,v) = \alpha(u) + \beta(v), \tag{2}$$

where α and β are arbitrary space curves of the parameters u and v (may be the arc-length parameters).

In this paper, we investigated the translation surfaces in 3-dimensional Euclidean space generated by two arbitrary space curves. Furthermore, a classification of flat and minimal translation surfaces has been obtained and some examples for geometric points on these surfaces were introduced.

2. Preliminaries

The Euclidean 3-dimensional space $\mathbf{E}^3$ is provided with the metric given by [10–12]:

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}^3$. Let $\delta : I \subset \mathbb{R} \to \mathbf{E}^3 : s \mapsto \delta(s)$ be an arbitrary curve in $\mathbf{E}^3$. The curve δ is said to be of unit speed (or parameterized by the arc-length parameter s) if $\langle \delta'(s), \delta'(s) \rangle = 1$ for any $s \in I$. Let $\{t(s), n(s), b(s)\}$ be the moving frame of δ, where the vectors t, n and b are the tangent, normal and binormal vectors, respectively. The Frenet equations for δ are given by

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}. \tag{3}$$

Let $S : \Phi = \Phi(u,v) \subset \mathbf{E}^3$ be a regular surface. Then the unit normal vector field of the surface S is given by

$$U = \frac{\Phi_u \wedge \Phi_v}{\|\Phi_u \wedge \Phi_v\|}, \quad \Phi_u = \frac{\partial \Phi(u,v)}{\partial u}, \quad \Phi_v = \frac{\partial \Phi(u,v)}{\partial v}, \tag{4}$$

where $\wedge$ stands the vector product of $\mathbf{E}^3$. The first fundamental form of the surface is induced from the metric of the ambient space $\mathbf{E}^3$

$$I = \langle d\Phi, d\Phi \rangle = E\, du^2 + 2F\, du\, dv + G\, dv^2, \tag{5}$$

with coefficients

$$E = \langle \Phi_u, \Phi_u \rangle, \quad F = \langle \Phi_u, \Phi_v \rangle, \quad G = \langle \Phi_v, \Phi_v \rangle.$$

Also, the second fundamental form of the surface S is given by

$$II = -\langle dU, d\Phi \rangle = L\, du^2 + 2M\, du\, dv + N\, dv^2, \tag{6}$$

where

$$L = \langle \Phi_{uu}, U \rangle, \quad M = \langle \Phi_{uv}, U \rangle, \quad N = \langle \Phi_{vv}, U \rangle.$$

Under this parametrization Φ, the Gauss and mean curvatures have the classical expressions, respectively

$$K = \frac{L\, N - M^2}{EG - F^2}, \tag{7}$$

$$H = \frac{E\, N + G\, L - 2F\, M}{2(EG - F^2)}. \tag{8}$$

The principal curvatures of the surface S are defined by

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K}. \tag{9}$$

In the light of the above, the sectional curvature κ_n and geodesic torsion τ_g are given by

$$\kappa_n = \frac{KH}{2H^2 - K}, \quad \tau_g = \pm \frac{K\sqrt{H^2 - K}}{2[2H^2 - K]}. \tag{10}$$

Now, we can write the following important definition:

Definition 2.1. A regular surface in $\mathbf{E}^3$ is a flat (developable) surface if $K = 0$ and a minimal surface if $H = 0$.

3. Curvatures on the translation surface

Let $X(u,v)$ be a translation surface in Euclidean 3-space $\mathbf{E}^3$ taking the form (2), where the variables u and v are the arc-length parameters for the two generating curves $\alpha(u)$ and $\beta(v)$, respectively. Let $\{t_\alpha, n_\alpha, b_\alpha\}$ be the Frenet frame field of the curve α with curvature κ_α and torsion τ_α. Also, let $\{t_\beta, n_\beta, b_\beta\}$ be the Frenet frame field of the curve β with curvature κ_β and torsion τ_β.

Calculating the partial derivative of (2) with respect to u and v respectively, we get

$$X_u = t_\alpha, \quad X_v = t_\beta, \tag{11}$$

since X_u and X_v are principal directions as any tangent vectors. From which, the components of the first fundamental form are

$$E = \langle t_\alpha, t_\alpha \rangle = 1, \quad F = \langle t_\alpha, t_\beta \rangle = \cos[\phi(u,v)],$$
$$G = \langle t_\beta, t_\beta \rangle = 1, \tag{12}$$

where $\phi(u,v)$ is the angle between tangent vectors of $\alpha(u)$ and $\beta(v)$. Then, the unit normal of the translation surface can be given by

$$U(u,v) = \frac{t_\alpha \wedge t_\beta}{\sin[\phi(u,v)]}, \quad \sin[\phi(u,v)] \neq 0. \tag{13}$$

Also, the components of the second fundamental form of X are obtained by

$$\begin{cases} L = \kappa_\alpha \cos[\theta_\alpha(u,v)], \\ M = 0, \\ N = \kappa_\beta \cos[\theta_\beta(u,v)], \end{cases} \tag{14}$$

where $\theta_\alpha(u,v)$ and $\theta_\beta(u,v)$ are the angles between U and n_α, n_β, respectively.

It is worth noting that: for the degenerate curves ($\kappa = 0$), the tangent, normal and binormal are constant vectors. From this note, it is easy to prove the following theorem:

Theorem 3.1. *For the translation surface (2), we have:*

(1) *The two variables u and v must appear in the angle ϕ between the tangent vectors t_α and t_β if and only if the two curves $\alpha(u)$ and $\beta(v)$ are non-degenerate curves ($\kappa_\alpha \neq 0$ and $\kappa_\beta \neq 0$).*
(2) *The variable u must only appear in the angle ϕ between the tangent vectors t_α and t_β if and only if $\kappa_\alpha \neq 0$ and $\kappa_\beta = 0$.*
(3) *The variable v must only appear in the angle ϕ between the tangent vectors t_α and t_β if and only if $\kappa_\alpha = 0$ and $\kappa_\beta \neq 0$.*
(4) *The angle ϕ between the tangent vectors t_α and t_β is constant if and only if the two curves $\alpha(u)$ and $\beta(v)$ are degenerate curves ($\kappa_\alpha = \kappa_\beta = 0$).*

From the above theorem, we can write the following lemma:

Lemma 3.2. *If the curvatures of the two generating curves of the translation surface (2) are vanished ($\kappa_\alpha = \kappa_\beta = 0$), then all angles $\theta_\alpha, \theta_\beta$ and ϕ are constants.*

Now, the principal curvatures κ_1 and κ_2, the Gaussian curvature K and the mean curvature H can be computed as the following:

$$\kappa_1 = \frac{\kappa_\alpha \cos[\theta_\alpha] + \kappa_\beta \cos[\theta_\beta]}{2\sin^2[\phi]}\left(1 + \left[1 - \frac{4\kappa_\alpha\kappa_\beta \cos[\theta_\alpha]\cos[\theta_\beta]\sin^2[\phi]}{(\kappa_\alpha \cos[\theta_\alpha] + \kappa_\beta \cos[\theta_\beta])^2}\right]^{1/2}\right), \tag{15}$$

$$\kappa_2 = \frac{\kappa_\alpha \cos[\theta_\alpha] + \kappa_\beta \cos[\theta_\beta]}{2\sin^2[\phi]}\left(1 - \left[1 - \frac{4\kappa_\alpha\kappa_\beta \cos[\theta_\alpha]\cos[\theta_\beta]\sin^2[\phi]}{(\kappa_\alpha \cos[\theta_\alpha] + \kappa_\beta \cos[\theta_\beta])^2}\right]^{1/2}\right), \tag{16}$$

$$K = k_1 k_2 = \frac{\kappa_\alpha\kappa_\beta \cos[\theta_\alpha]\cos[\theta_\beta]}{\sin^2[\phi]}, \tag{17}$$

$$H = \frac{k_1 + k_2}{2} = \frac{\kappa_\alpha \cos[\theta_\alpha] + \kappa_\beta \cos[\theta_\beta]}{2\sin^2[\phi]}. \tag{18}$$

From the Eq. (17), we can write the following:

Theorem 3.3. *If one of the generating curves is a degenerate curve, then the translation surface generated by two space curves is developable surface.*

Lemma 3.4. *The translation surface generated by two space curves in Euclidean 3-space is developable if and only if one of the following is satisfied:*

(1) *The angle ϕ is only a function of u.*
(2) *The angle ϕ is only a function of v.*
(3) *The angle ϕ is constant.*

Proof. The proof of parts (1)–(3) is resulted from parts (2)–(4) in theorem (3.1), respectively. □

From the Eq. (18), we can deduce the following theorem:

Theorem 3.5. *The translation surface X generated by two space curves in Euclidean 3-space is minimal surface if and only if the following condition is satisfied:*

$$\frac{\cos[\theta_\alpha(u,v)]}{\cos[\theta_\beta(u,v)]} = -\frac{\kappa_\beta(v)}{\kappa_\alpha(u)}. \tag{19}$$

On the other hand, if the translation surface is minimal, then the angle θ_β is given by:

$$\theta_\beta(u,v) = \cos^{-1}\left[-\frac{\kappa_\alpha(u)\cos[\theta_\alpha(u,v)]}{\kappa_\beta(v)}\right]. \tag{20}$$

Substituting the above equation in (17), we get

$$K = -\left(\frac{\kappa_\alpha \cos[\theta_\alpha]}{\sin[\phi]}\right)^2. \tag{21}$$

4. Classification of some points on translation surface

In this section, we will investigate how the normal curvature at a point on a surface varies when a unit tangent vector varies. It is known that a regular parameterized surface $X : \Omega \to \mathbf{E}^3$ (Ω is an open subset of the plane) has two principal curvatures $\kappa_1(u,v)$ and $\kappa_2(u,v)$ at each point $p = X(u,v)$ of the surface. If $\kappa_1(u,v) \leqslant \kappa_2(u,v)$ then $\kappa_1(u,v)$ is the minimum of normal curvatures in different directions at p, while $\kappa_2(u,v)$ is the maximum of them. If $\kappa_1(u,v) < \kappa_2(u,v)$ then the principal directions corresponding to κ_1 and κ_2 are uniquely defined, however if $\kappa_1 = \kappa_2$ then the normal curvature is constant in all directions and every direction is principal. Recalling that $EG - F^2$ is always strictly positive, we can classify some special points such as elliptic, hyperbolic, parabolic, singular as well as umbilical on the surface depending on the value of the Gaussian curvature K, and on the values of the principal curvatures κ_1 and κ_2 (or H). For the surface given by (2), we have the normal curvature κ_n as well as the geodesic torsion τ_g of that surface in the following forms

$$\kappa_n = \frac{\kappa_\alpha\kappa_\beta \cos[\theta_\alpha]\cos[\theta_\beta]\left(\kappa_\alpha \cos[\theta_\alpha] + \kappa_\beta \cos[\theta_\beta]\right)}{\kappa_\alpha^2 \cos^2[\theta_\alpha] + \kappa_\beta^2 \cos^2[\theta_\beta] + 2\kappa_\alpha\kappa_\beta \cos[\theta_\alpha]\cos[\theta_\beta]\cos^2[\phi]}, \tag{22}$$

$$\tau_g = \frac{\kappa_\alpha\kappa_\beta \cos[\theta_\alpha]\cos[\theta_\beta]\left[(\kappa_\alpha \cos[\theta_\alpha] + \kappa_\beta \cos[\theta_\beta])^2 - 4\kappa_\alpha\kappa_\beta \cos[\theta_\alpha]\cos[\theta_\beta]\sin^2[\phi]\right]^{\frac{1}{2}}}{2\left[\kappa_\alpha^2 \cos^2[\theta_\alpha] + \kappa_\beta^2 \cos^2[\theta_\beta] + 2\kappa_\alpha\kappa_\beta \cos[\theta_\alpha]\cos[\theta_\beta]\cos^2[\phi]\right]}. \tag{23}$$

Here, it is important to give the following definitions [13]:

Definition 4.1. Given a surface S, a point $p \in S$ belongs to one of the following kinds:

(1) Elliptic if $L\,N - M^2 > 0$, or equivalently $K > 0$.
(2) Hyperbolic if $L\,N - M^2 < 0$, equivalently $K < 0$.
(3) Parabolic if $L\,N - M^2 = 0$ and $L^2 + M^2 + N^2 > 0$ or equivalently $K = 0$ but either $\kappa_1 \neq 0$ or $\kappa_2 \neq 0$ i.e., only one from the principal curvatures equals zero.
(4) Umbilical point (or umbilic) if the principal curvatures at p are equal. An umbilical point p is said to be spherical if $\kappa_1 = \kappa_2 \neq 0$, and flat (planar) if $\kappa_1 = \kappa_2 = 0$.

Definition 4.2. A point p is an umbilical point if and only if $H^2 - K = 0$ at this point.

Definition 4.3. Let $\alpha(u)$ be a curve in Euclidean 3-space lying on a surface S. The following are well-known:

(1) $\alpha(u)$ is an asymptotic line if and only if the normal curvature κ_n vanishes.
(2) $\alpha(u)$ is a principal line if and only if the geodesic torsion τ_g vanishes.
(3) $\alpha(u)$ is a geodesic curve if and only if the geodesic curvature κ_g vanishes.

Now, according to (17), we investigate the geometric points on the translation surface (2), so let us distinguish the following cases:

Case (1) The point p is an *elliptic point* of the surface (2) if and only if $K > 0$. In this case $\kappa_\alpha \kappa_\beta \cos[\theta_\alpha]\cos[\theta_\beta] > 0$ and the principal curvatures of the surface k_1 and k_2 have the same sign. If κ_α and κ_β have the same sign, then θ_α and $\theta_\beta \in \left[0, \frac{\pi}{2}\right)$ or θ_α and $\theta_\beta \in \left(\frac{\pi}{2}, \pi\right]$. When κ_α and κ_β have opposite signs, then $\theta_\alpha \in [0, \frac{\pi}{2})$ and $\theta_\beta \in \left(\frac{\pi}{2}, \pi\right]$ or $\theta_\alpha \in \left(\frac{\pi}{2}, \pi\right]$ and $\theta_\beta \in \left[0, \frac{\pi}{2}\right)$.

Case (2) The point p is a *hyperbolic point* of the surface (2) if and only if $K < 0$. In this case $\kappa_\alpha \kappa_\beta \cos[\theta_\alpha]\cos[\theta_\beta] < 0$ and the principal curvatures of the surface k_1 and k_2 have opposite signs. If κ_α and κ_β have the same sign, then $\theta_\alpha \in \left[0, \frac{\pi}{2}\right)$ and $\theta_\beta \in \left(\frac{\pi}{2}, \pi\right]$ or $\theta_\alpha \in \left(\frac{\pi}{2}, \pi\right]$ and $\theta_\beta \in \left[0, \frac{\pi}{2}\right)$. When κ_α and κ_β have opposite signs, then θ_α and $\theta_\beta \in \left[0, \frac{\pi}{2}\right)$ or θ_α and $\theta_\beta \in \left(\frac{\pi}{2}, \pi\right]$.

Case (3) The point p is a *parabolic point* of the surface (2) if and only if $K = 0$. In this case, $\kappa_n = 0$ and $\tau_g = 0$. From (17), we have $\kappa_\alpha \kappa_\beta \cos[\theta_\alpha]\cos[\theta_\beta] = 0$. So that one of the following is satisfied: **(3.1):** $\kappa_\alpha = 0$ or $\kappa_\beta = 0$ or $\cos[\theta_\alpha]\cos[\theta_\beta] = 0$. If $\cos[\theta_\alpha] = 0$, then $\theta_\alpha = \frac{\pi}{2}(2\mathbf{n}+1), \mathbf{n} \in Z$, so $\alpha(u)$ is an asymptotic line as well as principal line. Similarly, if $\cos[\theta_\beta] = 0$, then $\theta_\beta = \frac{\pi}{2}(2\mathbf{n}+1), \mathbf{n} \in Z$, then $\beta(v)$ is also an asymptotic line.

Case (4) If $\cos[\theta_\alpha] = 0$ and $\cos[\theta_\beta] = 0$, then $\theta_\alpha = \frac{\pi}{2}(2\mathbf{n}+1)$ and $\theta_\beta = \frac{\pi}{2}(2\mathbf{n}+1)$, $\mathbf{n} \in Z$. In this case, the point p is *a planar point* of the surface (2) and $k_1 = k_2 = 0$.

Case (5) The point p is an *umbilical point* of the surface (2) if and only if $H^2 - K = 0$. From the Eqs. (17) and (18), we easily obtain

$$H^2 - K = \frac{1}{4\sin^4[\phi]}\Big[\kappa_\alpha^2\cos^2[\theta_\alpha] + \kappa_\beta^2\cos^2[\theta_\beta] + 2\kappa_\alpha\kappa_\beta\cos[\theta_\alpha]\cos[\theta_\beta](1 - 2\sin^2[\phi])\Big] = 0. \tag{24}$$

Based on the above equation, the point p is an *umbilical point* of the surface (2) if and only if one of the following is satisfied:

(I) $\kappa_\alpha = \kappa_\beta = 0$, which implies the Gaussian and mean curvatures are vanished. In this case, the point p is *a planar point* and $k_1 = k_2 = 0$.
(II) $\kappa_\alpha = 0$ and $\theta_\beta = \frac{\pi}{2}(2\mathbf{n}+1)$, $\mathbf{n} \in Z$, again we have $K = H = 0$ and the point p is again *a planar point*.
(III) $\kappa_\beta = 0$ and $\theta_\alpha = \frac{\pi}{2}(2\mathbf{n}+1)$, $\mathbf{n} \in Z$, one can get $K = H = 0$ and the point p is *a planar point too*.
(IV) $\theta_\alpha = \theta_\beta = \frac{\pi}{2}(2\mathbf{n}+1)$, $\mathbf{n} \in Z$, lead to $K = H = 0$.

As a consequence of the above cases, we give the following theorem:

Theorem 4.4. *If the surface (2) is not a developable surface, then there are not umbilical points on the translation surface (2) in Euclidean 3-space.*

Now, by considering Eqs. (2), (11) and (13), the singular points on the translation surface $X(u, v)$ are the points such that

$X_u \wedge X_v = 0,$

or equivalently

$\sin[\phi(u, v)] = 0,$

therefore, $\phi(u, v) = \mathbf{n}\pi, \mathbf{n} \in Z$. So, one can get the following theorems:

Theorem 4.5. *The point $X(u_0, v_0)$ of the translation surface $X(u, v)$ is a singular point if and only if*

$\sin[\phi(u, v)] = 0,$

Theorem 4.6. *If the point $X(u_0, v_0)$ of the translation surface $X(u, v)$ is a singular point, then we have:*

(1) The angle between tangent vectors of $\alpha(u)$ and $\beta(v)$ is equal to $\mathbf{n}\pi, \mathbf{n} \in Z$.
(2) The generated $\alpha(u)$ and $\beta(v)$ are degenerate curves, i.e., $\kappa_\alpha = \kappa_\beta = 0$.
(3) The point $X(u_0, v_0)$ of the translation surface $X(u, v)$ is an umbilical point.

5. Applications

We consider some important examples to illustrate the main results that we have presented in our paper.

Example 5.1. Let S be the translation surface which is not minimal and defined by (2) with generating two circular helices curves [8]:

$$\alpha(u) = \left(\sin\left[\frac{u}{2}\right], \cos\left[\frac{u}{2}\right] - 1, \frac{\sqrt{3}u}{2}\right), \quad \beta(v) = \left(\cos\left[\frac{v}{3}\right] - 1, \sin\left[\frac{v}{3}\right], \frac{2\sqrt{2}v}{3}\right). \tag{25}$$

The components of the first and second fundamental forms of this surface are given by, respectively:

$$E = 1, \quad F = \sqrt{\frac{2}{3}} - \frac{1}{6}\sin\left[\frac{3\ u + 2\ v}{6}\right], \quad G = 1, \tag{26}$$

$$L = \frac{4 + \sqrt{6}\sin\left[\frac{3\ u+2\ v}{6}\right]}{4\sqrt{23 + \cos\left[\frac{3\ u+2\ v}{3}\right] + 8\sqrt{6}\sin\left[\frac{3\ u+2\ v}{6}\right]}},$$
$$M = 0, \tag{27}$$
$$N = \frac{\sqrt{6} + 4\sin\left[\frac{3\ u+2\ v}{6}\right]}{9\sqrt{23 + \cos\left[\frac{3\ u+2\ v}{3}\right] + 8\sqrt{6}\sin\left[\frac{3\ u+2\ v}{6}\right]}}.$$

At the origin, we get the following:

$$E = G = 1, \quad F = \sqrt{\frac{2}{3}}, \quad L = \frac{1}{2\sqrt{6}}, \quad M = 0, \quad N = \frac{1}{18}. \tag{28}$$

$$\kappa_1 = \frac{2 + 3\sqrt{6} + \sqrt{58 + 4\sqrt{6}}}{24}, \quad \kappa_2 = \frac{2 + 3\sqrt{6} - \sqrt{58 + 4\sqrt{6}}}{24}, \tag{29}$$

$$\kappa_n = \frac{237 - 7\sqrt{6}}{4470}, \quad \tau_g = \frac{\sqrt{29 + 2\sqrt{6}}}{4\sqrt{3}(29 + 4\sqrt{6})}. \tag{30}$$

Since $K > 0$ and $L\ N - M^2 > 0$, the origin is an *elliptic point*.

Example 5.2. Scherk's Surface, obtained by H. Scherk in 1834, is the only non-flat minimal surface that can be presented as a translation surface of plane type [14]. Scherk's Surface is given by (2) with generating curves

$$\alpha(u) = \left(u, 0, \frac{1}{a}\log[\sec[a\ u]]\right), \quad \beta(v) = \left(0, v, \frac{1}{a}\log[\cos[a\ v]]\right). \tag{31}$$

For the two generating curves (31), we have

$$\begin{cases} t_\alpha = (\cos[a\ u], 0, \sin[a\ u]), & t_\beta = (0, \cos[a\ v], -\sin[a\ v]), \\ n_\alpha = (-\sin[a\ u], 0, \cos[a\ u]), & n_\beta = -(0, \sin[a\ v], \cos[a\ v]) \\ b_\alpha = (0, -1, 0), & b_\beta = (-1, 0, 0) \\ \kappa_\alpha = a\cos[a\ u], & \kappa_\beta = a\cos[a\ v]. \end{cases} \tag{32}$$

Then, the unit normal vector is given by

$$U(u, v) = \frac{1}{\sin[\phi(u, v)]}(-\sin[a\ u]\cos[a\ v], \cos[a\ u] \times \sin[a\ v], \cos[a\ u]\cos[a\ v]), \tag{33}$$

where

$$\sin[\phi(u, v)] = \sqrt{1 - \sin^2[a\ u]\sin^2[a\ v]}. \tag{34}$$

From the above, we can obtain the following:

$$\cos[\theta_\alpha(u, v)] = \frac{\cos[a\ v]}{\sqrt{1 - \sin^2[a\ u]\sin^2[a\ v]}}, \quad \cos[\theta_\beta(u, v)] = \frac{-\cos[a\ u]}{\sqrt{1 - \sin^2[a\ u]\sin^2[a\ v]}}. \tag{35}$$

On the other hand, the components E, F and G of the first fundamental form and L, M and N of the second fundamental form are given by

$$\begin{aligned} &E = \sec^2[a\ u], \quad F = -\tan[a\ u]\tan[a\ v], \quad G = \sec^2[a\ v], \\ &L = \frac{a\sec^2[a\ u]}{\sqrt{\sec^2[a\ v] + \tan^2[a\ u]}}, \quad M = 0, \quad N = \frac{-a\sec^2[a\ v]}{\sqrt{\sec^2[a\ v] + \tan^2[a\ u]}}. \end{aligned} \tag{36}$$

Hence, the Gaussian curvature of the Scherk's surface is:

$$K = -\left(\frac{a\cos[a\ u]\cos[a\ v]}{1 - \sin^2[a\ u]\sin^2[a\ v]}\right)^2. \tag{37}$$

The mean curvature is vanished, i.e., $H = 0$.

Remark 5.1. For the minimal surface generated by the two plane curves (31) and from the above results we can write the following:

(1) Eq. (34) of the two variables u and v appeared in the angle ϕ is harmonic with Part (1) in theorem (3.1).

(2) The results in the Eqs. (32) and (35) are harmonic with lemma (3.4) and the main theorem (3.5) of minimal surface.

The principal curvatures as well as the normal curvature and the geodesic torsion of Scherk's surface are given by, respectively:

$$\begin{aligned} &\kappa_1 = -\kappa_2 = \frac{a\cos[a\ u]\cos[a\ v]}{1 - \sin^2[a\ u]\sin^2[a\ v]}, \quad \kappa_n = 0, \\ &\tau_g = -\frac{a\cos[a\ u]\cos[a\ v]}{2(1 - \sin^2[a\ u]\sin^2[a\ v])}. \end{aligned} \tag{38}$$

At the origin, we have

$$\begin{aligned} &E = G = 1, \quad F = 0, \quad L = -N = a, \quad M = 0, \\ &K = -a^2, \quad \kappa_1 = -\kappa_2 = a, \quad \kappa_n = 0, \quad \tau_g = -\tfrac{a}{2}. \end{aligned} \tag{39}$$

Since, the Gaussian curvature is negative at the origin, then the origin is a *hyperbolic point*.

Example 5.3. Let the translation surface S be the cylinder defined by (2) such that one of the generating curves is a circle with radius a and the other is a straight-line perpendicular to the plane of the circle. These two curves can be written as:

$$\alpha(u) = (a\cos[u], a\sin[u], 0), \quad \beta(v) = (0, 0, v). \tag{40}$$

For these curves, one can easily get the following:

$$\begin{aligned} &E = a^2, \quad F = 0, \quad G = 1, \quad L = -a, \quad M = N = 0, \\ &K = 0, \quad H = -\frac{1}{2\ a}, \quad \kappa_1 = 0, \quad \kappa_2 = -\frac{1}{a}, \quad \kappa_n = \tau_g = 0. \end{aligned} \tag{41}$$

because $L\ N - M^2 = 0$ and $L^2 + M^2 + N^2 > 0$, then, all points on the cylinder are *parabolic points*.

Example 5.4. Let S be a translation surface given by (2) with generating two plane curves lying on parallel planes and given by [15]:

$$\begin{aligned} &\alpha(u) = \int \left(\cos\left[\int \kappa_\alpha(u)\ du\right], \sin\left[\int \kappa_\alpha(u)\ du, 0\right]\right) du, \\ &\beta(v) = \int \left(\cos\left[\int \kappa_\beta(v)\ dv\right], \sin\left[\int \kappa_\beta(v)\ dv\right], 0\right) dv, \end{aligned} \tag{42}$$

where κ_α and κ_β are the curvatures of the two curves $\alpha(u)$ and $\beta(v)$, respectively. By straightforward computations, one can obtain the following:

$$\begin{aligned} &E = 1, \quad F = \cos\left[\int \kappa_\alpha(u)\ du + \int \kappa_\beta(v)\ dv\right], \quad G = 1, \\ &L = M = N = K = H = \kappa_1 = \kappa_2 = 0. \end{aligned} \tag{43}$$

In this case $U_u = U_v = 0$, therefore the normal vector is constant along the surface. The derivative of the function $\langle U, X\rangle$ with respect to u and v is $\langle U, X_u\rangle = \langle U, X_v\rangle = 0$ because U is perpendicular to the tangent vectors X_u, X_v, hence $\langle U, X\rangle$ is constant and the surface is contained in a plane. This means that all points of the surface S are planar points while this surface has not umbilical points.

Example 5.5. Let S be a surface in Euclidean 3-space, then S is written as

$$X(u, v) = \alpha(u) + \beta(v), \tag{44}$$

with generated curves

$$\alpha(u) = (u^3, 0, 0), \quad \beta(v) = (0, v^3, 0), \quad u \in [-1, 1], \ v \in [-1, 1].$$

Calculating the partial derivative of (44) with respect to u and v respectively, we get

$$\begin{cases} X_u = (3u^2, 0, 0), \\ X_v = (0, 3v^2, 0). \end{cases}$$

Then

$$X_u \wedge X_v = (0, 0, 9u^2 v^2).$$

At the origin, by a straightforward computations, one can get the following:

$$X_u \wedge X_v = (0, 0, 0).$$

Since $X_u \wedge X_v = 0$, hence the origin is a singular point.

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